Existence of fixed points for a particular multifunction

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Abstract. In this paper, we prove that if E is an infinite-dimensional reflexive real Banach space possessing the Kadec-Klee property, then, for every compact function from the unit sphere S of E to the dual E^* satisfying the condition $\inf_{x\in S}\|f(x)\|_{E^*} > 0$, there exists $\hat{x} \in S$ such that

$$f(\hat{x})(\hat{x}) = \|f(\hat{x})\|_{E^*} \ .$$

Key words and phrases: Kadec-Klee property, reflexivity, upper semicontinuous multifunction, fixed point, Fan-Kakutani theorem.

2010 Mathematics Subject Classification: 46B10, 46B20, 47H04, 47H10.

Here and in the sequel, E is a reflexive real Banach space, with dual E^*. Set

$$S = \{x \in E : \|x\| = 1\} \ .$$

Let $f : S \to E^*$ be a continuous function.

In this very short paper, we are interested in the existence of some $\hat{x} \in S$ such that

$$f(\hat{x})(\hat{x}) = \|f(\hat{x})\|_{E^*} \ .$$

Clearly, if, for each $x \in S$, we set

$$\Phi_f(x) := \{y \in S : f(x)(y) = \|f(x)\|_{E^*}\} \ ,$$

our problem is equivalent to finding a fixed point of the multifunction Φ_f.

Note that, by reflexivity, we have $\Phi_f(x) \neq \emptyset$ for all $x \in S$. However, the mere continuity of f is not enough to guarantee the existence of solutions to our problem.

Let us recall that, when $\dim(E) = \infty$, E is said to have the Kadec-Klee property if for every sequence $\{x_n\}$ in S weakly converging to $x \in S$, one has $\lim_{n\to\infty} \|x_n - x\| = 0$.

Also, we say that a function $\psi : S \to E^*$ is compact if it is continuous and $\psi(S)$ is relatively compact.

Here is our contribution about the above problem.

THEOREM 1. - *Let E be infinite-dimensional and have the Kadec-Klee property, and let $f : S \to E^*$ be a compact function such that*

$$\inf_{x\in S} \|f(x)\|_{E^*} > 0 \ . \tag{1}$$

Then, there exists $\hat{x} \in S$ such that

$$f(\hat{x})(\hat{x}) = \|f(\hat{x})\|_{E^*} \ .$$

Let us recall that a multifunction $F : X \to 2^Y$ between two topological spaces is said to be upper semicontinuous if, for each closed set $C \subseteq Y$, the set

$$F^-(C) := \{x \in X : F(x) \cap C \neq \emptyset\}$$

is closed in X.

Proof of Theorem 1. Consider the multifunction $\Psi : E^* \to 2^S$ defined by putting

$$\Psi(\varphi) = \{x \in S : \varphi(x) = \|\varphi\|_{E^*}\}$$

for all $\varphi \in E^*$. Let us show that the restriction of Ψ to $E^* \setminus \{0\}$ is upper semicontinuous. To this end, let $C \subseteq S$ be a (non-empty) closed set and let $\{\varphi_n\}$ be a sequence in $\Psi^-(C) \setminus \{0\}$ converging in E^* to $\varphi \neq 0$. We have to show that $\varphi \in \Psi^-(C)$. For each $n \in \mathbf{N}$, choose $x_n \in C$ so that

$$\varphi_n(x_n) = \|\varphi_n\|_{E^*} \ . \tag{2}$$

By reflexivity, there is a subsequence $\{x_{n_k}\}$ weakly converging to some $x \in E$. Reading (2) with n_k instead of n and passing to the limit for $k \to \infty$, we get

$$\varphi(x) = \|\varphi\|_{E^*} \ .$$

Since $\varphi \neq 0$, we have $x \in S$. Consequently, since E has the Kadec-Klee property, $\{x_{n_k}\}$ converges strongly to x. Therefore, since C is closed, we have $x \in C$. Hence, $x \in \Psi(\varphi) \cap C$ and so $\varphi \in \Psi^-(C)$. Next, observe that, for each $\varphi \in E^* \setminus \{0\}$, the set $\Psi(\varphi)$ is bounded, closed and convex, and so it is weakly compact, by reflexivity. But, since E has the Kadec-Klee property, each weakly compact subset of S is compact, in view of the Eberlein-Smulyan theorem. So, each set $\Psi(\varphi)$, with $\varphi \neq 0$, is compact. Next, note that, by (1), $K := \overline{f(S)}$ is a compact set in E^* which does not contain 0. Consequently, by the upper semicontinuity of $\Psi_{|(E^*\setminus\{0\})}$, the set $\Psi(K)$ is compact ([2], Theorem 7.4.2). Since

$\dim(E) = \infty$, there is a continuous function $\omega : B \to S$ such that $\omega(x) = x$ for all $x \in S$, where B is the closed unit ball of E. Finally, denote by Y the closed convex hull of $\Psi(K)$ and set

$$G(x) = \Psi(f(\omega(x)))$$

for all $x \in Y$. So, G is an upper semicontinuous multifunction (as the composition of the upper semicontinuous multifunction $\Psi_{|(E^* \setminus \{0\})}$ and the continuous function $f \circ \omega$) with non-empty, closed and convex values, from the compact convex set Y into itself. Then, by the Fan-Kakutani theorem ([1]), there exists $\hat{x} \in Y$ such that $\hat{x} \in G(\hat{x})$. Then, since $\hat{x} \in S$, we have $\omega(\hat{x}) = \hat{x}$, and so $\hat{x} \in \Phi_f(\hat{x})$, as desired. $\triangle$

Some remarks about the assumptions of Theorem 1 are now in order.

Assume that $(E, \langle \cdot, \cdot, \rangle)$ is a Hilbert space. Consider the continuous function $f : S \to E^*$ defined by putting

$$f(x)(y) = -\langle x, y \rangle$$

for all $x \in S$, $y \in E$. In this case, we have

$$f(x)(x) = -1$$

and

$$\|f(x)\|_{E^*} = 1$$

for all $x \in S$. This example shows, at the same time, that Theorem 1 is no longer true if either the infinite dimensionality of E or the compactness of f is removed.

Also, note that, concerning the compactness of f, a more sophisticated example can be provided in any infinite-dimensional Banach space. Actually, in this case, E. Michael ([3]) proved that there exists a continuous function $f : S \to E^*$ such that

$$f(x)(x) = 0$$

and

$$\|f(x)\|_{E^*} = 1$$

for all $x \in S$.

Concerning the necessity of (1), consider the following example.

Let E be the space of all absolutely continuous functions $u : [0, 1] \to \mathbf{R}$ with $u' \in L^2([0, 1])$. In other words, let E be the usual Sobolev space $H^1(0, 1)$, with the usual norm

$$\|u\| = \left(\int_0^1 |u'(t)|^2 dt + \int_0^1 |u(t)|^2 dt \right)^{\frac{1}{2}} .$$

Consider the continuous function $f : S \to E^*$ defined by putting

$$f(u)(v) = -\int_0^1 u(t)v(t)dt$$

for all $u \in S$, $v \in E$. Note that E has the Kadec-Klee property since it is a Hilbert space. Moreover, since E is compactly embedded into $C^0([0,1])$, the function f is compact. Finally, we have

$$f(u)(u) = -\int_0^1 |u(t)|^2 dt < 0 < \int_0^1 |u(t)|^2 dt \leq \|f(u)\|_{E^*}$$

for all $u \in S$.

We conclude by proposing the following problem.

PROBLEM 1. - Let E be an infinite-dimensional reflexive real Banach space such that, for each compact function $f : S \to E^*$ satisfying (1), there exists $\hat{x} \in S$ for which

$$f(\hat{x})(\hat{x}) = \|f(\hat{x})\|_{E^*} \ .$$

Then, does E possess the Kadec-Klee property ?

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